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Dual equivalence in models with higher-order derivatives

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Abstract

We introduce a class of higher-order derivative models in (2, 1) spacetime dimensions. The models are described by a vector field, and contain a Proca-like mass term which prevents gauge invariance. We use the gauge embedding procedure to generate another class of higher-order derivative models, gauge-invariant and dual to the former class. We show that the results are valid in arbitrary $(d, 1)$ spacetime dimensions when one discards the Chern–Simons and Chern–Simons-like terms. We also investigate duality at the quantum level, and we show that it is preserved in the quantum scenario. Other results include investigations concerning the gauge embedding approach when the vector field couples with fermionic matter, and when one adds nonlinearity.

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1. Introduction

It is hardly necessary to recall the remarkable and powerful properties of the duality mapping as an analytical tool in field theory as well as in string theory [1]. On the other hand, the interest in the study of theories involving higher-order derivatives is by now well appreciated and remains intense. Within the context of Maxwell theory, generalizations involving higher-order derivatives can be found in [2–4]. More recently, in [5, 6] one finds generalizations that involve both the Maxwell and Chern–Simons (CS) terms [7–9]. In the present work we study duality symmetry in extended theories, which contain higher-order derivatives involving both the Maxwell and CS terms.

The interest in the subject has been recently fuelled by several motivations, in particular by issues related to string theory. As one knows, string theories engender the feature of containing interactions that may involve an infinite number of spacetime derivatives, and that may lead to standard field theory in the low energy limit. Thus, higher-order derivative contributions would certainly appear when one consider next-to-leading orders in the energy. Based upon such interesting possibilities, there is a renewed interest in investigating higher-order derivative models, including the case of tachyons [10–17]. Other lines of investigations can be found for instance in [18, 19], and also in gravity, where higher orders in the scalar curvature R are considered in the so-called nonlinear gravity theories—see, for instance, the recent works [20, 21] and references therein.

The presence of higher-order derivatives to control the behaviour of systems is not peculiar to string and field theories. It may also appear in condensed matter, and can be important to describe higher-order phase transitions [22], such as for instance the case of the fourth-order transition in superconducting $Ba_{0.6}K_{0.4}BiO_3$, which is described in [23]. Furthermore, sometimes one has to include higher-order derivatives of the order parameter to correctly describe pattern formation in chemical reactions, and in other branches of nonlinear science [24, 25].

We investigate the subject dealing with issues that appear very naturally in field theory. Specifically, we examine the duality mapping of higher-derivative extensions of self-dual (SD) and Maxwell–Chern–Simons (MCS) theories, including the presence of fermionic matter. We work in the (2, 1)-dimensional spacetime, with $\varepsilon^{012} = \varepsilon_{012} = 1$; our metric tensor has signature $(+, -, -)$ and we use natural units. We start in the next section 2, where we examine the dual mapping of the MCS–Proca model with the MCS–Podolsky theory. Our methodology makes use of the gauge embedding procedure, an approach which has been shown to work very efficiently to unveil the dual partner of a specific model [26–30]. The investigation of the MCS–Proca model is new, and we use it to set the stage to generalize the model to higher-order derivatives. We develop this generalization in section 3, where we introduce the main model, which is defined in terms of $n = 1, 2, \ldots,$ and in section 4, where we consider the case $n \to \infty$. We show in section 3 that if one discards the CS and CS-like terms, the results are then valid in arbitrary $(d, 1)$ spacetime dimensions. We also show in section 4 that the duality is preserved at the quantum level. In section 5 we examine the presence of matter, coupling fermions to the system. In section 6 we change the model introduced in section 3 to include the presence of nonlinearity. We end our work in section 7, where we present our comments and conclusions.

2. Duality transformation in the MCS–Proca model

It is well known that both the SD and the MCS models are dual representations of the same dynamics: they carry one massive degree of freedom of definite helicity determined by the relative sign of the CS term. However, the SD representation hides a gauge symmetry, which is explicit in the MCS model. This is easily seen when one establishes the correspondence $f_\mu \to F_\mu \sim \varepsilon_{\mu\nu\rho}\partial^\nu A^\rho$, which maps the SD field f_μ into the dual of the basic field A_μ of the MCS model. We note that in the above dual mapping, one relates a gauge non-invariant model with an equivalent, gauge invariant theory. In this process, however, the non-gauge field is identified with a special form containing the derivative of the gauge field of the dual model. The identification of this mechanism is crucial for the generalizations that we will implement below.

In this section we elaborate on a prototype of the study we intend develop in this paper. We discuss the theoretical motivations for studying higher derivatives dualities, present a physical scenario for the applications of the ideas elaborated and the quantum implications.

In the past, there have appeared several different ways of extending electrodynamics, trying to smooth infrared or ultraviolet singularities that appear at large or short distances. One is the Born–Infeld [2] type of generalization, and others include the generalizations introduced by Proca [3] and by Podolsky [4]. The Born–Infeld approach involves nonlinear extension, which will be considered in section 6. The Proca model involves the addition of a mass term for the vector field, which was introduced to smooth infrared singularities. The Podolsky model involves higher-order derivatives and was introduced to smooth ultraviolet singularities. Thus, the Proca and Podolsky models deal with dual aspects of the electromagnetic interaction, and so they may be connected by some duality procedure.

The dual aspects of electrodynamics that appear in the Proca and Podolsky models constitute the central subject of the present section. This duality is also of importance in the extension of the bosonization programme from $D = 2$ to higher dimensions. The mechanisms of the 3D bosonization in particular are quite dependent on the duality results involving the presence of the Chern–Simons term. Besides providing us with the proper scenario for the applications of our ideas, bosonization will also be crucial for the interpretation of the new parameters in the Podolsky extension.

2.1. The duality procedure

In order to exemplify the general procedure of duality, we consider the more general model

$$\mathcal{L} = \frac{m^2}{2} A^\mu A_\mu - \frac{a}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} m \varepsilon_{\mu\nu\lambda} A^\mu \partial^\nu A^\lambda \tag{1}$$

where m is the mass parameter. We have introduced the real and dimensionless parameter a in order to obtain the SD model for $a = 0$, or the MCS–Proca model for $a = 1$.

The equation of motion involves second-order derivatives, and the model is supposedly dual to the generalized MCS–Podolsky theory. To verify this assumption we follow the gauge embedding procedure [26–30] to construct its dual equivalent model. Firstly, we compute the Euler vector associated with the MCS–Proca theory. We get

$$K_\mu = m^2 A_\mu - m \varepsilon_{\mu\nu\lambda} \partial^\nu A^\lambda + a \partial^\nu F_{\nu\mu}. \tag{2}$$

The first iteration leads to

$$\mathcal{L}_1 = \mathcal{L}_0 - K^\mu B_\mu \tag{3}$$

where $\mathcal{L}_0$ is identified with the MCS–Proca Lagrangian given by equation (1). Also, B_μ is an auxiliary field, which varies according to

$$\delta B_\mu = \delta A_\mu = \partial_\mu \Lambda. \tag{4}$$

This choice makes the non-invariant term in $\mathcal{L}_0$ cancel with the term $K_\mu \delta B^\mu$. Therefore

$$\delta \mathcal{L}_1 = -B_\mu \delta K^\mu = -\frac{m^2}{2} \delta(B^2). \tag{5}$$

Thus, we can write the gauge invariant second-iterated Lagrangian

$$\mathcal{L}_2 = \mathcal{L}_0 - K^\mu B_\mu + \frac{m^2}{2} B^2. \tag{6}$$

This ends the iteration process, and we can eliminate the auxiliary field to obtain the dual model

$$\mathcal{L}_D = \mathcal{L}_0 - \frac{K^2}{2m^2} \tag{7}$$

or better

$$\mathcal{L}_D = \frac{a-1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m \varepsilon_{\mu\nu\lambda} A^\mu \partial^\nu A^\lambda - \frac{a^2}{2m^2} \partial_\mu F^{\mu\nu} \partial^\lambda F_{\lambda\nu} + \frac{a}{m} \varepsilon^{\mu\nu\lambda} \partial_\nu A_\lambda \partial^\rho F_{\rho\mu}. \tag{8}$$

This is the generalized MCS–Podolsky model—see [4–6] for further information on this and other related models. We recall that the Podolsky model was introduced in order to smooth ultraviolet singularities. In this sense, we note that our approach for duality is working standardly, since we are linking infrared and ultraviolet problems, in MCS–Proca and MCS–Podolsky models.

In the above investigation, if $a = 0$ the MCS–Proca model in (1) becomes the standard SD model; in this case, in the generalized MCS–Podolsky model (8) one kills the Podolsky terms, and we are led back to the MCS model. On the other hand if $a = 1$ we get to an extended CS–Podolsky model.

It seems important, at this juncture, to establish the scenario at which the parameter a becomes a physical quantity. This is done next where we show that action (8) for the generalized MCS–Podolsky model is the low energy effective action for a self-interaction fermionic model. To set the problem in a proper perspective, we review shortly the bosonization procedure.

2.2. Physical interpretation

It is well known by now that the bosonization in $D = 3$ maps a massive scalar particle coupled to a Chern–Simons gauge field into a massive Dirac fermion for a special value of the Chern–Simons coupling. This is a relevant issue in the context of transmutation of spin and statistics with interesting applications to problems both in quantum field theory and condensed matter physics. This boson–fermion transmutation is a property which holds only at very long distances, namely at scales long compared with the Compton wavelength of the particle. Thus these results hold to the lowest order in an expansion in powers of the inverse mass of the particle.

The equivalence of the three-dimensional effective electromagnetic action of the CP^1 model with a charged massive fermion to the lowest order in inverse (fermion) mass has been proposed by Deser and Redlich [31]. Using the results of [31], bosonization was extended from two to three dimensions [32]. However, contrary to the two-dimensional case where the fermionic determinant can be exactly computed, bosonization in higher dimensions is not exact and, in the general case, it has a non-local structure. However, for the large mass limit in the one-loop of perturbative evaluation, a local expression materializes. Indeed, there has been established [32], to leading order in the inverse fermionic mass, an identity between the partition functions for the three-dimensional Thirring model and the topologically massive $U(1)$ gauge theory, whose dynamics is controlled by a Maxwell–Chern–Simons action. Here we show that the contribution next-to-leading order leads to the MCS–Podolsky model via duality transformation.

Below we compute the low energy sector of a theory of massive self-interacting fermions, the massive Thirring model in 2 + 1 dimensions, that can be bosonized into a gauge theory, the Maxwell–Chern–Simons gauge theory and its possible higher derivative extensions. We start from the fermionic partition function for the three-dimensional massive Thirring model, following closely the spirit of [32],

$$\mathcal{Z}_{\mathrm{Th}} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \, \exp\left(-\int \left(\bar{\psi}^i(\partial\!\!\!/ + M)\psi^i - \frac{g^2}{2N} j^\mu j_\mu\right) \mathrm{d}^3x\right) \tag{9}$$

with the coupling constant g^2 having dimensions of inverse mass. Here ψ^i are N two-component Dirac spinors and j^μ is a global $U(1)$ current defined as,

$$j^\mu = \bar{\psi}^i \gamma^\mu \psi^i. \tag{10}$$

Note that we have reverted to the Euclidean metric in this subsection. As usual, we eliminate the quartic interaction by performing a (functional) Legendre transformation through the identity

$$\exp\left(\int \frac{g^2}{2N} j^\mu j_\mu \, \mathrm{d}^3x\right) = \int \mathcal{D}A_\mu \, \exp\left(-\int \left(\frac{1}{2}A^\mu A_\mu + \frac{g}{\sqrt{N}} j^\mu A_\mu\right) \mathrm{d}^3x\right) \tag{11}$$

after a scaling $A_\mu \to g A_\mu/\sqrt{N}$. The partition function then becomes

$$\mathcal{Z}_{\mathrm{Th}} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A_\mu \, \exp\left(-\int \left(\bar{\psi}^i \left(\partial\!\!\!/ + M + \frac{g}{\sqrt{N}} A\!\!\!/\right)\psi^i + \frac{1}{2}A^\mu A_\mu\right) \mathrm{d}^3x\right). \tag{12}$$

Formally, the fermionic path-integral gives the Dirac operator determinant,

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \, \exp\left(-\int \bar{\psi}^i \left(\partial\!\!\!/ + M + \frac{g}{\sqrt{N}} A\!\!\!/\right)\psi^i \, \mathrm{d}^3x\right) = \det\left(\partial\!\!\!/ + M + \frac{g}{\sqrt{N}} A\!\!\!/\right). \tag{13}$$

The determinant of the Dirac operator is an unbounded operator and requires regularization. Bosonization will depend on the actual computation of this determinant, namely whether it can be computed exactly in a closed form or an approximate recipe must be enforced. In general it leads to non-local structures but, under some approximation scheme (such as the inverse mass), a local result emerges.

This determinant can be computed exactly for $D = 2$, both for Abelian and non-Abelian symmetries. For the $D = 3$ case this determinant has been computed in [31] as an expansion in inverse powers of the fermion mass giving, in the leading order, the Chern–Simons parity violating term, as well as the parity conserving Maxwell term, which is central to our discussion here,

$$\ln\det\left(\partial\!\!\!/ + M + \frac{g}{\sqrt{N}} A\!\!\!/\right) = \pm\frac{\mathrm{i}g^2}{16\pi}\int \epsilon_{\mu\nu\alpha} F^{\mu\nu} A^\alpha \, \mathrm{d}^3x - \frac{g^2}{24\pi M}\int \mathrm{d}^3x \, F^{\mu\nu}F_{\mu\nu} + O(\partial^2/M^2). \tag{14}$$

We bring these results into the partition function to get

$$\mathcal{Z}_{\mathrm{Th}} = \int \mathcal{D}A_\mu \, \mathrm{e}^{-\mathcal{S}_{\mathrm{eff}}[A_\mu]} \tag{15}$$

where $\mathcal{S}_{\mathrm{eff}}[A_\mu]$ is given, up to order $1/m$, by

$$\mathcal{S}_{\mathrm{eff}}[A_\mu] = \frac{1}{2}\int \mathrm{d}^3x \left(A_\mu A^\mu \mp \frac{\mathrm{i}g^2}{4\pi}\epsilon^{\mu\alpha\nu} A_\mu \partial_\alpha A_\nu - \frac{g^2}{12\pi M} F^{\mu\nu}F_{\mu\nu}\right) \tag{16}$$

after the scaling $A_\mu \to m A_\mu$, with $m = 4\pi/g^2$. With these identifications we find $a = \frac{2m}{3M}$. In conclusion, we have established the following identification:

$$\mathcal{Z}_{\mathrm{Th}} \approx \mathcal{Z}_{\mathrm{MCS-Proca}} \tag{17}$$

which is valid to next-to-leading order in $1/M$.

In the preceding section we have established the dynamical equivalence between the model defined by $\mathcal{S}_{\mathrm{MCS-Proca}}$ and the MCS–Podolsky theory. This proves the equivalence, to this order in $1/M$ expansion, of the partition functions for the Thirring model and the MCS–Podolsky theory:

$$\mathcal{Z}_{\mathrm{Th}} \approx \mathcal{Z}_{\mathrm{MCS-Podolsky}} \tag{18}$$

It expresses the equivalence between three-dimensional interacting fermionic theory and Maxwell–Chern–Simons vectorial bosons in the long wavelength approximation. It is interesting to observe that the Thirring coupling constant g^2/N in the fermionic model is mapped into the inverse mass spin 1 massive excitation. In the same fashion, we may identify each term of the generalization with higher-order derivatives studied below, with terms following from the $1/M$ expansion of the fermionic determinant. This is however beyond the scope of the present work.

3. Generalization to higher-order derivatives

In this section we use the duality procedure which connects the SD and MCS models, and the MCS–Proca and MCS–Podolsky models, in order to extend the formalism to generalized models involving higher-order derivatives. Evidently, the presence of higher-order derivatives introduces longer distance effects and is of interest for string theory, and for investigations involving phase transition. We split the subject into two subsections, the first exploring the relevant classical issues, and the second dealing with duality at the quantum level.

3.1. Duality at the classical level

Let us rescale fields and coordinates as $A \to m^{1/2}\bar{A}$ and $x \to m^{-1}\bar{x}$, in order to work with dimensionless quantities. We rewrite $\bar{A}$ and $\bar{x}$ as A and x again, and we define the general field $A_\mu^{(n)}$ through the recursive relation

$$A^{(n)\mu} \equiv \epsilon^{\mu\nu\rho}\partial_\nu A_\rho^{(n-1)}. \tag{19}$$

Here we use $A_\mu^{(0)} = A_\mu$ to represent the basic field. The subscript n which identifies the field also shows the number of derivatives one has to perform in the basic field. We note that in the above relation (19) the field $A_\mu^{(n)}$ is related to the field $A_\mu^{(n-1)}$ by means of a derivative, which maintains the mechanism we have identified in section 2, where the dual relation between the fields involves a derivative.

For this field we now define the general field strength,

$$F_{\mu\nu}^{(n)} \equiv \partial_\mu A_\nu^{(n)} - \partial_\nu A_\mu^{(n)}. \tag{20}$$

With this, the Maxwell term is proportional to $A_\mu^{(1)}A^{(1)\mu}$, and can be generalized to $A_\mu^{(n)}A^{(n)\mu}$.

We define the antisymmetric tensor $G_{\mu\nu}^{(n)}$ such that $G_{\mu\nu}^{(0)} = F_{\mu\nu}$, and

$$G_{\mu\nu}^{(n)} \equiv \partial_{[\mu}\partial^\lambda G_{\lambda\nu]}^{(n-1)} = \partial_\mu\partial^\lambda G_{\lambda\nu}^{(n-1)} - \partial_\nu\partial^\lambda G_{\lambda\mu}^{(n-1)} \tag{21}$$

which is important to relate $A_\mu^{(n)}$ to $A_\mu^{(0)}$. We note that $G_{\mu\nu}^{(n)}$ contains $2n+1$ implicit derivatives in A_μ. So, the general field is, for n positive,

$$A_\mu^{(n)} = \begin{cases} (-1)^{\frac{n}{2}}\partial^\nu G_{\nu\mu}^{(n/2-1)} & \text{for} \quad n \text{ even} \\ \frac{1}{2}(-1)^{\frac{n-1}{2}}\varepsilon_\mu^{\ \nu\lambda}G_{\nu\lambda}^{(n/2-1/2)} & \text{for} \quad n \text{ odd.} \end{cases} \tag{22}$$

To include CS-like terms into the proposed generalization we first note that the CS term can be written as $\varepsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda = A_\mu^{(0)}A^{(1)\mu}$. Thus, it induces the general form, $A_\mu^{(i)}A^{(j)\mu}$, where i, j are non-negative integers. We note that if $i = j$ we obtain the generalized Maxwell term. Thus, we can modify the generalized Maxwell model appropriately, to include extended CS contributions. Furthermore, we can prove the identity

$$A_\mu^{(i)}A^{(j)\mu} = A_\mu^{(i-1)}A^{(j+1)\mu} + \varepsilon^{\mu\nu\lambda}\partial_\mu\left(A_\nu^{(i-1)}A_\lambda^{(j)}\right). \tag{23}$$

The last term is a total derivative, which can be discarded since it does not change the action when this identity is used in the Lagrange density of the corresponding model.

We see that for $i + j$ even we can write, apart from a total derivative,

$$A_\mu^{(i)}A^{(j)\mu} = A_\mu^{(\frac{i+j}{2})}A^{(\frac{i+j}{2})\mu}. \tag{24}$$

As a result, in the action we do not need to split $A_\mu^{(i)}A^{(j)\mu}$ into generalized Maxwell and extended CS terms. We introduce a single term, characterized by the sum $i + j$: for $i + j$ even we get a generalized Maxwell term, and for $i + j$ odd we obtain an extended CS term. For example, $A_\mu^{(0)}A^{(1)\mu}$ and $A_\mu^{(0)}A^{(2)\mu}$ reproduce the standard CS and Maxwell terms, respectively.

We use the above results to introduce the model

$$\mathcal{L}^{(n)} = \sum_{i=0}^{n} r_i A_\mu^{(0)} A^{(i)\mu} \tag{25}$$

where n is integer, and r_i are real and dimensionless parameters, with $r_0 \neq 0$. The equation of motion is

$$\sum_{i=0}^{n} r_i A_\mu^{(i)} = 0. \tag{26}$$

This equation allows us to write $\partial^\mu A_\mu^{(0)} = 0$, which shows that no longitudinal mode propagates in the theory. Besides, we can use this condition to rewrite equation (22) in the simpler form

$$A_\mu^{(n)} = \begin{cases} (-1)^{\frac{n}{2}} \Box^{\frac{n}{2}} A_\mu^{(0)} & \text{for} \quad n \text{ even} \\ (-1)^{\frac{n-1}{2}} \Box^{\frac{n-1}{2}} \varepsilon_{\mu\nu\lambda} \partial^\nu A^{(0)\lambda} & \text{for} \quad n \text{ odd.} \end{cases} \tag{27}$$

With the aim to build an Abelian gauge model, we consider the variation $\delta A_\mu = \delta A_\mu^{(0)} = \partial_\mu \Lambda$; as before, Λ is a local infinitesimal parameter. We use this and the definition for $A_\mu^{(n)}$ to obtain

$$\delta A_\mu^{(i)} = 0 \qquad i \neq 0. \tag{28}$$

We vary the Lagrange density in (25) to obtain

$$\delta \mathcal{L}^{(n)} = K^\mu \delta A_\mu^{(0)} \tag{29}$$

where the local Noether current is defined as

$$K_\mu \equiv 2 \sum_{i=0}^{n} r_i A_\mu^{(i)}. \tag{30}$$

We introduce an auxiliary vector field a_μ, and we couple it linearly to the Euler vector

$$\mathcal{L}_1^{(n)} = \mathcal{L}^{(n)} - a^\mu K_\mu. \tag{31}$$

We chose $\delta a_\mu = \partial_\mu \Lambda = \delta A_\mu$. Thus

$$\delta \mathcal{L}_1^{(n)} = -a_\mu \delta K^\mu. \tag{32}$$

We consider

$$\mathcal{L}_2^{(n)} = \mathcal{L}_1^{(n)} + r_0 a^\mu a_\mu. \tag{33}$$

The procedure ends with the elimination of the auxiliary field a_μ. We get to

$$\mathcal{L}_D^{(n)} = \mathcal{L}^{(n)} - \frac{1}{4r_0} K^\mu K_\mu \tag{34}$$

which is the Lagrange density of the dual model. We use the Euler vector to get

$$\mathcal{L}_D^{(n)} = \sum_{i=0}^{n} r_i A_\mu^{(0)} A^{(i)\mu} - \frac{1}{r_0} \sum_{i=0}^{n} \sum_{j=0}^{n} r_i r_j A_\mu^{(i)} A^{(j)\mu} \tag{35}$$

or better

$$\mathcal{L}_D^{(n)} = -\frac{1}{r_0} \sum_{i=1}^{n} \sum_{j=0}^{n} r_i r_j A_\mu^{(i)} A^{(j)\mu} \tag{36}$$

which gives the dual model. We note that the respective action is gauge-invariant. This result shows that the model (25) engenders hidden gauge invariance, in a way similar to the SD model [7]. We can rewrite this result in the form

$$\mathcal{L}_D = \sum_{i=1}^{2n} r_i' A_\mu^{(0)} A^{(i)\mu} \tag{37}$$

where we have used

$$r_i' = \begin{cases} -\frac{1}{r_0} \sum_{l=1}^{i} r_l r_{i-l} & \text{for} \quad 1 \leqslant i \leqslant n \\ -\frac{1}{r_0} \sum_{l=i-n}^{n} r_l r_{i-l} & \text{for} \quad n < i \leqslant 2n. \end{cases} \tag{38}$$

Before ending this section, let us comment on issues related to the above duality investigation. The equation of motion that follows from the dual model (37) is

$$\sum_{i=1}^{2n} r_i' A_\mu^{(i)} = 0. \tag{39}$$

In equation (26) we eliminate $A_\mu^{(0)}$ to see that these two equations are identical, thus confirming the dual equivalence between the two theories.

Another issue concerns finding a master theory. To get to this, let us introduce another field $B_\mu^{(i)}$ such that $B_\mu^{(0)} = B_\mu$. We use $A_\mu^{(i)}$ and $B_\mu^{(i)}$ to write

$$\mathcal{L}_M^{(n)} = r_0 B_\mu^{(0)} B^{(0)\mu} + 2 \sum_{i=1}^{n} r_i B_\mu^{(0)} A^{(i)\mu} - \sum_{i=1}^{n} r_i A_\mu^{(0)} A^{(i)\mu}. \tag{40}$$

We vary the corresponding action with respect to $B^{(0)}$ to get

$$B^{(0)} = -\frac{1}{r_0} \sum_{i=1}^{n} r_i A^{(i)\mu}. \tag{41}$$

We use this in (40) to obtain

$$\mathcal{L}_D^{(n)} = -\frac{1}{r_0} \sum_{i=1}^{n} \sum_{j=0}^{n} r_i r_j A_\mu^{(i)} A^{(j)\mu} \tag{42}$$

which is equation (36). If we vary the master action with respect to $A_\mu^{(0)}$ to get

$$\sum_{i=1}^{n} r_i B_\mu^{(i)} = \sum_{i=1}^{n} r_i A_\mu^{(i)}. \tag{43}$$

This result allows us to write

$$\mathcal{L} = r_0 B_\mu^{(0)} B^{(0)\mu} + \sum_{i=1}^{n} r_i B_\mu^{(0)} B^{(i)\mu} = \sum_{i=0}^{n} r_i B_\mu^{(0)} B^{(i)\mu} \tag{44}$$

which reproduces equation (25), confirming that equation (40) is a master or parent theory.

The model that we have introduced is defined by equation (25); it is controlled by the integer n which we choose to be $n = 1, 2, 3, \ldots$. The simplest case is $n = 1$, which reproduces the SD model. The next case is $n = 2$, which gives the MCS–Proca model investigated in section 2. The case $n = 3$ gives

$$\begin{aligned} \mathcal{L}^{(3)} &= r_0 A_\mu^{(0)} A^{(0)\mu} + r_1 A_\mu^{(0)} A^{(1)\mu} + r_2 A_\mu^{(0)} A^{(2)\mu} + r_3 A_\mu^{(0)} A^{(3)\mu} \\ &= r_0 A_\mu A^\mu + r_1 \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + \tfrac{1}{2} r_2 F_{\mu\nu} F^{\mu\nu} + r_3 \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu \partial^\rho F_{\lambda\rho} \end{aligned} \tag{45}$$

and the other cases follow in standard fashion. For $n = 3$ the dual model is

$$\mathcal{L}_D^{(3)} = -\frac{1}{r_0}\sum_{i=1}^{3}\sum_{j=0}^{3} r_i r_j A_\mu^{(i)} A^{(j)\mu}. \tag{46}$$

We use equations (21) and (22) to write

$$\begin{aligned}\mathcal{L}_D^{(3)} = {} & -\frac{r_1}{2}\varepsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda} - \frac{1}{2}\left(r_2 + \frac{r_1^2}{r_0}\right) F_{\mu\nu} F^{\mu\nu} \\ & + \frac{1}{2}\left(r_3 + 2\frac{r_1 r_2}{r_0}\right)\varepsilon^{\mu\nu\lambda} F_{\mu\nu}\partial^\rho F_{\rho\lambda} - \frac{1}{r_0}\left(r_2^2 + 2r_1 r_3\right)\partial_\mu F^{\mu\nu}\partial^\lambda F_{\lambda\nu} \\ & - 2\frac{r_2 r_3}{r_0}\varepsilon^{\mu\nu\lambda}\partial^\rho F_{\rho\mu}\partial_\nu\partial^\alpha F_{\alpha\lambda} - 2\frac{r_3^2}{r_0}\partial^\mu\partial^\nu F_{\nu\lambda}\partial_\mu\partial_\rho F^{\rho\lambda}.\end{aligned} \tag{47}$$

In this example, we can choose the parameters $r_0 \neq 0, r_1, r_2, r_3$ to find the dual theory to every model, up to the order $n = 3$. For instance, if we use $r_0 = 1/2, r_1 = -1/2, r_2 = -a/2$, and $r_3 = 0$, and if we re-introduce dimensional units, we obtain from (45) and (47) expressions (1) and (8) that we have investigated in section 2.

We note that the presence of the CS and CS-like terms imposes the restriction that the Minkowski space must have (2, 1) spacetime dimensions. Thus, if we set $r_i = 0$ for i odd, we eliminate all the CS and CS-like terms, and our results are then valid in arbitrary $(d, 1)$ spacetime dimensions.

3.2. Quantum duality

We now explore the issue of quantum duality [33–35] for the models examined previously. We start with the Lagrange density for the master theory, given by equation (40). We extend the model to write

$$\mathcal{L}_M^{(n)}(J, j) = r_0 B_\mu^{(0)} B^{(0)\mu} + 2B_\mu^{(0)} F^\mu(A) - A_\mu^{(0)} F^\mu(A) + \alpha r_0 B_\mu^{(0)} j^{(0)\mu} + \alpha F^\mu(A) J^{(0)\mu} \tag{48}$$

where we have set

$$F_\mu(A) = \sum_{i=1}^{n} r_i A_\mu^{(i)} \tag{49}$$

$$F_\mu(B) = \sum_{i=1}^{n} r_i B_\mu^{(i)} \tag{50}$$

to ease calculation. We see that the master theory contains two fields, thus we have added the external currents j_μ and J_μ; we see that j_μ couples with B_μ, while J_μ couples with A_μ indirectly, through $F_\mu(A)$.

We use (48) to eliminate A_μ. We obtain $F_\mu(A) = F_\mu(B) + (\alpha/2)F_\mu(J)$, which we use to write

$$\mathcal{L}^{(n)}(J, j) = \mathcal{L}^{(n)} + \alpha B_\mu^{(0)}(r_0 j^{(0)\mu} + F^\mu(J)) + \frac{\alpha^2}{4} J_\mu^{(0)} F^\mu(J) \tag{51}$$

where $\mathcal{L}^{(n)}$ is the model given by equation (25). We also use (48) to eliminate $B_\mu^{(0)}$. We get $B_\mu^{(0)} = -(1/r_0)F_\mu^{(0)}(A) - (\alpha/2)j_\mu^{(0)}$, which we use to obtain

$$\mathcal{L}_D^{(n)}(J, j) = \mathcal{L}_D^{(n)} + \alpha F_\mu(A)(J^{(0)\mu} - j^{(0)\mu}) - \frac{\alpha^2 r_0}{4} j_\mu j^\mu \tag{52}$$

where $\mathcal{L}_D^{(n)}$ is the dual model, given by equation (36).

We define the functional generators $Z_M^{(n)}(J, j)$, $Z^{(n)}(J, j)$ and $Z_D^{(n)}(J, j)$ in the usual way. They allow us to obtain

$$\frac{1}{Z_M^{(n)}} \frac{\delta^2 Z_M^{(n)}}{\delta j_\mu(x)\delta j_\nu(y)}\bigg|_{j,J=0} = \frac{1}{Z^{(n)}} \frac{\delta^2 Z^{(n)}}{\delta j_\mu(x)\delta j_\nu(y)}\bigg|_{j,J=0} = -\alpha^2 r_0^2 \langle B^{(0)\mu}(x) B^{(0)\nu}(y)\rangle \tag{53}$$

and

$$\frac{1}{Z_M^{(n)}} \frac{\delta^2 Z_M^{(n)}}{\delta J_\mu(x)\delta J_\nu(y)}\bigg|_{j,J=0} = \frac{1}{Z_D^{(n)}} \frac{\delta^2 Z_D^{(n)}}{\delta J_\mu(x)\delta J_\nu(y)}\bigg|_{j,J=0} = -\alpha^2 \langle F^\mu(A)(x) F^\nu(A)(x)\rangle_D. \tag{54}$$

The master theory allows us to write

$$\frac{1}{Z_M^{(n)}} \frac{\delta^2 Z_M^{(n)}}{\delta j_\mu(x)\delta j_\nu(y)}\bigg|_{j,J=0} = \frac{1}{Z_D^{(n)}} \frac{\delta^2 Z_D^{(n)}}{\delta j_\mu(x)\delta j_\nu(y)}\bigg|_{j,J=0}$$
$$= -\alpha^2 \langle F^\mu(A)(x) F^\nu(A)(y)\rangle_D - \mathrm{i}\frac{\alpha^2 r_0}{2}\delta^{\mu\nu}\delta(x-y) \tag{55}$$

and

$$\frac{1}{Z_M^{(n)}} \frac{\delta^2 Z_M^{(n)}}{\delta J_\mu(x)\delta J_\nu(y)}\bigg|_{j,J=0} = \frac{1}{Z^{(n)}} \frac{\delta^2 Z^{(n)}}{\delta J_\mu(x)\delta J_\nu(y)}\bigg|_{j,J=0}$$
$$= -\alpha^2 \langle F^\mu(B)(x) F^\nu(B)(y)\rangle + \mathrm{i}\frac{\alpha^2}{2} O^{\mu\nu}\delta(x-y) \tag{56}$$

where the operator $O^{\mu\nu}$ is defined according to

$$F^\mu(A) = O^{\mu\nu} A_\nu^{(0)}. \tag{57}$$

We use these results to write

$$\left\langle B_\mu^{(0)}(x) B_\nu^{(0)}(y)\right\rangle = \langle F_\mu(A)(x) F_\nu(A)(y)\rangle_D + \mathrm{i}\frac{r_0}{2}\delta_{\mu\nu}\delta(x-y) \tag{58}$$

and

$$\langle F^\mu(B)(x) F^\nu(B)(y)\rangle - \mathrm{i}\frac{1}{2} O^{\mu\nu}\delta(x-y) = \langle F^\mu(A)(x) F^\nu(A)(y)\rangle_D. \tag{59}$$

Expressions (58) and (59) show that the corresponding Green functions are equivalent, apart from contact terms. They show that the duality that we have presented in section 3.1 above is preserved at the quantum level.

We now examine the specific case considered in section 2, which involves duality between the MCS–Proca theory (1) and the generalized MCS–Podolsky model (8). In this case the results (58) and (59) become

$$\langle B_\mu(x) B_\nu(y)\rangle = \langle F_\mu(A)(x) F_\nu(A)(y)\rangle_D + \mathrm{i}\frac{m^2}{4}\delta_{\mu\nu}\delta(x-y) \tag{60}$$

and

$$\langle F_\mu(B)(x) F_\nu(B)(y)\rangle - \frac{\mathrm{i}}{2}\left(-\frac{m}{2}\varepsilon_{\mu\nu\lambda}\partial^\lambda + \frac{a}{2}\delta_{\mu\nu}\Box - \frac{a}{2}\partial_\mu\partial_\nu\right)\delta(x-y)$$
$$= \langle F_\mu(A)(x) F_\nu(A)(y)\rangle_D \tag{61}$$

where we have re-inserted the parameters used in section 2. We see that for $a = 0$ one recovers the result of [34], which deals with quantum duality for the CS and MCS models.

4. The case $n \to \infty$

The success of the former investigations has led us to think of extending the case containing a finite number of terms to the case where an infinity sequence of terms is considered. Our interest relies on the case where the infinite sequence of terms adds to give elementary functions. We see that the form of $\mathcal{L}^{(n)}$ in (25) suggests this particular generalization, to the case where $n \to \infty$. Such a generalization can be written in terms of a smooth function, depending on the specific values of the real parameters r_i that we have introduced to define the model. We further explore this possibility introducing the model

$$\mathcal{L} = A_\mu [F(\mathcal{O})]^\mu_\nu A^\nu \tag{62}$$

where F is a smooth function, and $\mathcal{O}$ is the operator

$$\mathcal{O}^{\mu\nu} \equiv -\varepsilon^{\mu\nu\lambda}\partial_\lambda. \tag{63}$$

The model is defined in terms of the expansion of the smooth function, in the form

$$\mathcal{L} = A_\mu \sum_{n=0}^{\infty} C_n [\mathcal{O}^n]^\mu_\nu A^\nu \tag{64}$$

where C_n are given by

$$C_n = \frac{1}{n!}\frac{\mathrm{d}^n F(x)}{\mathrm{d}x^n}\Big|_{x=0} \tag{65}$$

and $[\mathcal{O}^0]^\mu_\nu = \delta^\mu_\nu$, $[\mathcal{O}^1]^\mu_\nu = -\varepsilon^\mu_{\nu\lambda}\partial^\lambda$ and $[\mathcal{O}^2]^\mu_\nu = [\mathcal{O}^1]^\mu_\lambda [\mathcal{O}^1]^\lambda_\nu$, and so forth. We require that $F(0) \neq 0$, which implies that $C_0 \neq 0$. This means that the above model starts with the Proca-like term $A_\mu A^\mu$, which prevents the presence of gauge invariance. Thus, we use the gauge embedding procedure to obtain the dual model. It has the form

$$\mathcal{L}_D = A_\mu \sum_{n=0}^{\infty} C_n [\mathcal{O}^n]^\mu_\nu A^\nu - \frac{1}{C_0} A_\mu \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_n C_m [\mathcal{O}^{n+m}]^\mu_\nu A^\nu \tag{66}$$

or, formally,

$$\mathcal{L}_D = A_\mu \left[F(\mathcal{O}) - \frac{1}{F(0)} [F(\mathcal{O})]^2 \right]^\mu_\nu A^\nu. \tag{67}$$

We illustrate this general result with the example

$$\mathcal{L} = A_\mu \left(\frac{1}{1+\mathcal{O}} \right)^\mu_\nu A^\nu. \tag{68}$$

In this case the dual theory reads

$$\mathcal{L}_D = A_\mu \left(\frac{\mathcal{O}}{(1+\mathcal{O})^2} \right)^\mu_\nu A^\nu. \tag{69}$$

5. Adding fermions

The study of fermions is motivated by the possibility of extending the present duality procedure to more realistic models, which should necessarily contain fermionic matter fields to describe the matter content of any realistic model.

We add fermions with the modification

$$\tilde{\mathcal{L}}^{(n)} = \mathcal{L}^{(n)} + \mathcal{L}_I + \mathcal{L}_f \tag{70}$$

where

$$\mathcal{L}_f = \bar{\psi}(\mathrm{i}\partial\!\!\!/ - M)\psi \tag{71}$$

describes free fermions, with M being the dimensionless fermion mass parameter. To identify how the fermionic field interacts with the fields $A_\mu^{(n)}$ we introduce the non-minimal coupling

$$\partial_\mu \to D_\mu = \partial_\mu + \mathrm{i}\sum_{i=0}^{n} e_i A_\mu^{(i)} \tag{72}$$

where e_i are (dimensionless) coupling constants—we note that the condition $e_i = 0, i \neq 0$ leads to minimal coupling with the fermionic matter. With the above generic coupling, the interaction terms in $\mathcal{L}_I$ are given by

$$\mathcal{L}_I = -\sum_{i=0}^{n} e_i A^{(0)\mu} J_\mu^{(i)} \tag{73}$$

where we have defined the general current

$$J_\mu^{(i)} \equiv \varepsilon_{\mu\nu\lambda}\partial^\nu J^{(i-1)\lambda} \tag{74}$$

with

$$J_\mu^{(0)} \equiv j_\mu = \bar{\psi}\gamma_\mu\psi. \tag{75}$$

To write the above expressions we have used the identity

$$A_\mu^{(i)} J^{(j)\mu} = A_\mu^{(i-1)} J^{(j+1)\mu} + \varepsilon^{\mu\nu\lambda}\partial_\mu\big[A_\nu^{(i-1)} J_\lambda^{(j)}\big]. \tag{76}$$

We now search for the dual theory. The procedure follows as in the former case. The Euler vector is modified by the presence of interactions; it changes to

$$\tilde{K}_\mu = K_\mu - \sum_{i=0}^{n} e_i J_\mu^{(i)}. \tag{77}$$

We introduce an auxiliary field $\tilde{a}_\mu$, and we impose that $\delta\tilde{a}_\mu = \delta A_\mu^{(0)} = \delta\Lambda$. The Lagrange density varies according to $\delta\mathcal{L}^{(1)} = -\tilde{a}_\mu\delta\tilde{K}^\mu$, since $\delta J_\mu^{(n)} = 0$, as one can verify straightforwardly. As in the former case, the procedure requires another iteration. The final result is

$$\tilde{\mathcal{L}}_D^{(n)} = \tilde{\mathcal{L}}^{(n)} - \frac{1}{4r_0}\tilde{K}^\mu\tilde{K}_\mu \tag{78}$$

or, explicitly,

$$\tilde{\mathcal{L}}_D^{(n)} = -\sum_{i=1}^{2n}\left(r_i' A_\mu^{(0)} A^{(i)\mu} - s_i' A_\mu^{(0)} J^{(i)\mu}\right) - \frac{1}{4}\sum_{i=0}^{2n} e_i' J_\mu^{(0)} J^{(i)\mu} + \mathcal{L}_f \tag{79}$$

where we have set

$$s_i' = \begin{cases} -\frac{1}{r_0}\sum_{l=1}^{i} r_l e_{i-l} & \text{for} \quad 1 \leqslant i \leqslant n \\ -\frac{1}{r_0}\sum_{l=i-n}^{n} r_l e_{i-l} & \text{for} \quad n < i \leqslant 2n \end{cases} \tag{80}$$

and

$$e_i' = \begin{cases} \frac{1}{r_0}\sum_{l=0}^{i} e_l e_{i-l} & \text{for} \quad 0 \leqslant i \leqslant n \\ \frac{1}{r_0}\sum_{l=i-n}^{n} e_l e_{i-l} & \text{for} \quad n < i \leqslant 2n. \end{cases} \tag{81}$$

In this result, we note the presence of Thirring-like interactions [36], which are fundamental to maintain the contents of the fermionic sector unchanged [35]. To see this we examine the dynamics of the fermionic sectors in both theories. The fermionic equation of motion for the first theory is

$$(\mathrm{i}\partial\!\!\!/ - M)\psi = \sum_{i=0}^{n} e_i A_\mu^{(i)} \gamma^\mu \psi. \tag{82}$$

To eliminate the gauge field, we note that

$$\sum_{i=0}^{n} r_i A_\mu^{(i)} = \frac{1}{2} \sum_{i=0}^{n} e_i J_\mu^{(i)}. \tag{83}$$

We restrict the coupling constants to obey $e_i = \alpha r_i$ to get

$$(\mathrm{i}\partial\!\!\!/ - M)\psi = \frac{1}{2}\alpha \sum_{i=0}^{n} e_i J_\mu^{(i)} \gamma^\mu \psi. \tag{84}$$

Analogously, the fermionic equation for the dual theory is given by

$$(\mathrm{i}\partial\!\!\!/ - M)\psi = \sum_{i=1}^{2n} s'_i A_\mu^{(i)} \gamma^\mu \psi + \frac{1}{2} \sum_{i=1}^{2n} e'_i J_\mu^{(i)} \gamma^\mu \psi. \tag{85}$$

We find the equation of motion for the gauge field in the dual model as

$$\sum_{i=1}^{2n} r'_i A_\mu^{(i)} = \frac{1}{2}\alpha \sum_{i=1}^{2n} s'_i J_\mu^{(i)}. \tag{86}$$

The restriction $e_i = \alpha r_i$ implies that $s'_i = \alpha r'_i$, and so we can write

$$(\mathrm{i}\partial\!\!\!/ - M)\psi = \frac{1}{2}\alpha \sum_{i=0}^{n} e_i J_\mu^{(i)} \gamma^\mu \psi \tag{87}$$

which is equation (84). This result shows that the fermionic sector does not change when one goes from (70) to the dual theory (79).

We can also add fermions in the case $n \to \infty$. To illustrate this possibility we consider the model

$$\tilde{\mathcal{L}} = A_\mu [F(\mathcal{O})]^\mu_\nu A^\nu - A_\mu [G(\mathcal{O})]^\mu_\nu J^\nu + \mathcal{L}_f \tag{88}$$

where G is a smooth function similar to F. We use the gauge embedding procedure to obtain the dual model

$$\begin{aligned}\tilde{\mathcal{L}}_D = A_\mu \left[F(\mathcal{O}) - \frac{1}{F(0)} F^2(\mathcal{O}) \right]^\mu_\nu A^\nu - A_\mu \left[G(\mathcal{O}) - \frac{1}{F(0)} F(\mathcal{O}) G(\mathcal{O}) \right]^\mu_\nu J^\nu \\ - \frac{1}{4} J_\mu \left[\frac{1}{F(0)} G^2(\mathcal{O}) \right]^\mu_\nu J^\nu + \mathcal{L}_f.\end{aligned} \tag{89}$$

We note the presence of the generalized Thirring-like term in the dual theory.

6. Nonlinear interactions

We now return to the Born–Infeld [2] generalization, which is different from the Proca and Podolsky generalizations. The main ingredient now is nonlinearity, and so we further explore the duality procedure introduced above mixing higher-order derivatives and nonlinearity.

Evidently, there are several different possibilities of including nonlinearity in the model introduced in section 3, but here we consider the case

$$\mathcal{L}_{\rm NL}^{(n)} = g\big(A_\mu^{(0)} A^{(0)\mu}\big) + \sum_{i=1}^{n} r_i A_\mu^{(0)} A^{(i)\mu} \tag{90}$$

where $g(x)$ is nonlinear in $x = A_\mu^{(0)} A^{(0)\mu}$. This case does not include nonlinear interactions that involve derivatives of the basic field $A_\mu = A_\mu^{(0)}$. The equation of motion is

$$A_\mu^{(0)} = -\frac{1}{g'(x)} \sum_{i=1}^{n} r_i A_\mu^{(i)} \tag{91}$$

where $g'(x) = \mathrm{d}g/\mathrm{d}x$. This equation allows us to write

$$\partial^\mu A_\mu^{(0)} = -\sum_{i=1}^{n} r_i A_\mu^{(i)} \partial^\mu \left(\frac{1}{g'(x)}\right) \tag{92}$$

which shows that there are longitudinal modes propagating, due to the presence of the nonlinear interaction. We note that in the linear case $[g(x) = x]$ we have $g' = 1$, which leaves no room for propagation of longitudinal modes.

We treat the presence of nonlinearity invoking the trick used in [28], and then further explored in [29, 37]. The key point here is to remove the nonlinearity at the expense of introducing another field, an auxiliary scalar field ϕ. We implement this possibility with the change

$$g\big(A_\mu^{(0)} A^{(0)\mu}\big) \rightarrow f(\phi) + \frac{1}{\phi} A_\mu^{(0)} A^{(0)\mu}. \tag{93}$$

We follow [37] to show that

$$f(\phi) = \int^\phi \mathrm{d}\chi \, \frac{1}{\chi^2} g'^{-1}\left(\frac{1}{\chi}\right). \tag{94}$$

The model is modified to

$$\mathcal{L}_\phi^{(n)} = f(\phi) + \frac{1}{\phi} A_\mu^{(0)} A^{(0)\mu} + \sum_{i=1}^{n} r_i A_\mu^{(0)} A^{(i)\mu}. \tag{95}$$

In this case the Euler vector is given by

$$K_\mu = \frac{2}{\phi} A_\mu^{(0)} + 2\sum_{i=1}^{n} r_i A_\mu^{(i)}. \tag{96}$$

The gauge embedding method allows us to write

$$\mathcal{L}_D^{(n)} = \mathcal{L}_\phi^{(n)} - \tfrac{1}{4}\phi K_\mu K^\mu \tag{97}$$

and so

$$\mathcal{L}_D^{(n)} = f(\phi) - \sum_{i=1}^{n} r_i A_\mu^{(0)} A^{(i)\mu} - \phi \sum_{i=2}^{2n} \tilde{r}_i A_\mu^{(0)} A^{(i)\mu} \tag{98}$$

where $\tilde{r}_i$ is given by

$$\tilde{r}_i = \begin{cases} \sum_{l=1}^{i-1} r_l r_{i-l} & \text{for} \quad 2 \leqslant i \leqslant n \\ \sum_{l=i-n}^{n} r_l r_{i-l} & \text{for} \quad n < i \leqslant 2n. \end{cases} \tag{99}$$

We note that in (98) the nonlinear behaviour involves all the terms, except for the Chern–Simons one. This fact is clearer when we eliminate the auxiliary field ϕ from the model, which is formally given by

$$\phi = f'^{-1}\left(\sum_{i=2}^{2n} \tilde{r}_i A_\mu^{(0)} A^{(i)\mu}\right). \tag{100}$$

We illustrate the above investigations with the model

$$\mathcal{L}_{\mathrm{BI}}^{(n)} = \beta^2 \sqrt{1 + \frac{2r_0}{\beta^2} A_\mu^{(0)} A^{(0)\mu}} + \sum_{i=1}^{n} r_i A_\mu^{(0)} A^{(i)\mu} \tag{101}$$

where the nonlinear contribution is of Born–Infeld type. We note that the limit $\beta \to \infty$ restores the original model (25). In this case, the function $f(\phi)$ given by equation (94) has the form

$$f(\phi) = \frac{1}{2}\beta^2 \left(r_0\phi + \frac{1}{r_0\phi}\right). \tag{102}$$

We use equation (100) to obtain

$$\phi = \frac{1}{r_0} \Bigg/ \sqrt{1 - \frac{2}{r_0\beta^2} \sum_{i=2}^{2n} \tilde{r}_i A_\mu^{(0)} A^{(i)\mu}}\ . \tag{103}$$

The gauge embedding procedure given above allows us to write

$$\mathcal{L}_{\mathrm{BID}}^{(n)} = \beta^2 \sqrt{1 - \frac{2}{r_0\beta^2} \sum_{i=2}^{2n} \tilde{r}_i A_\mu^{(0)} A^{(i)\mu}} - \sum_{i=1}^{n} r_i A_\mu^{(0)} A^{(i)\mu}. \tag{104}$$

We see that in the limit $\beta \to \infty$ the above result leads to (37), the dual of the model (25), as expected. We also note that in the case $n = 1$ the model (101) reproduces the Born–Infeld–Chern–Simons model investigated in [28], and this shows that the model (101) is a generalization of the model investigated in [28].

7. Comments and conclusions

In the present work we have investigated duality symmetry in generalized field theory models, involving higher-order derivatives of the basic field $A_\mu = A_\mu^{(0)}$ through the recursive relation $A_\mu^{(n)} = \varepsilon_\mu^{\ \nu\lambda}\partial_\nu A_\lambda^{(n-1)}$. The investigations started in section 2 and in section 3, and there we have generalized the self-dual model to include several higher-order derivative terms, and we have obtained the dual theory. We have also proposed a master model, from which one gets both the model and its dual partner. In this generalization, the presence of the CS and CS-like terms imposes the restriction that the Minkowski space has (2, 1) spacetime dimensions. However, we can eliminate all the CS and CS-like terms with the restriction $r_i = 0$ for i odd; in this case our results are valid in arbitrary $(d, 1)$ spacetime dimensions. In section 3 we have also examined the implications of duality at the quantum level. These results are obtained for $\mathcal{L}^n$, for n integer, finite, and in section 4 we have further extended our results, considering the limit $n \to \infty$, which leads to the case involving non-polynomial functions.

Later, in section 5 we have added fermions to the system, and there we have included fermions in models involving $A_\mu^{(n)}$, in the case where n is finite, and also for $n \to \infty$. In section 6 we have investigated the more general case, where nonlinear contributions involving the basic field $A_\mu^{(0)}$ are present. As we have shown, in this case we first circumvent nonlinearity at the expense of introducing an auxiliary field, and then we proceed as before, to get to the dual model. After getting to the dual model, we then eliminate the auxiliary field to obtain the dual model in terms of the original field $A_\mu^{(0)}$ and the accompanying derivatives.

We note that all the generalizations we have investigated involve the Abelian vector field A_μ. Thus, a natural and direct issue concerns the use of non-Abelian fields, and their possible generalizations, along the lines of the non-Abelian SD model, and the Yang–Mills–Chern–Simons model [26]. Also, it is of interest to investigate if the presence of bosonic matter changes the duality scenario that we have presented in section 5. Another point concerns duality in models of the $B \wedge F$ type, and their extensions to include higher-order derivatives. These and other related issues are under consideration, and we hope to report on them in the near future. Another point concerns models which include fermions. As one knows, if there is no fermionic self-interaction, and if one integrates on the fermions, the remaining action will necessarily contain higher-order derivative terms, thus giving another compelling motivation to investigate models involving higher-order derivatives. We hope to report on these and other related issues in another work.

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